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AUTHOR Shaw, Doris J.; And Others

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ERROR ANALYSIS OF FRACTION ARITHMETIC - SELECTED CASE STUDIES

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DORIS J. SHAW
SALLY N. STANDIFORD
MARY F. KLEIN
KIKUMI K. TATSUOKA

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Copies of this report may be requested from:

Kikumi Tatsuoka
103 S. Mathews
252 ERL (Education Research Lab)
University of Illinois at Urbana-Champaign
Urbana, IL 61801

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Abstract

This report describes the test performances of 26 students who displayed a variety of erroneous rules in completing a 48-item test of fraction addition problems, constructed to diagnose at least 88 projected erroneous rules. Cases were selected for their potential usefulness in designing and implementing an error-diagnostic testing system and in developing appropriate remedial instruction.

Each case describes specific examples of mislearning, information obtained through teacher-student interviews, remedial instruction, and the results from a posttest. The report concludes with several conjectures about the sources and causes of some of the misconceptions.

Introduction

A 48-item test of fraction addition problems that can diagnose at least 88 projected erroneous rules described in Klein, et al., (1981) was administered to junior high students. The free responses to each item in the open-ended test were subjected to an extensive error analysis. This report describes the test performances of students who used a variety of erroneous rules either partially or throughout the test. One of the authors of the report (who is a teacher of these students) gave individualized remedial instruction in order to remedy her students' misconceptions and administered the same test afterward. Some misconceptions persisted for three or four months until the teacher pointed out the errors and corrected them. Other misconceptions disappeared by the time the teacher tried to correct them.

The intention of this report is to write up the cases which will be useful in constructing an error-diagnostic testing system, as well as preparing remedial instruction based on the test results. Of course, we have tried to describe the information obtained from the interviews with the students and the error analyses of test performances as objectively as possible, but there are still some portions of this report that are speculation, especially in the part describing the sources and causes of some misconceptions. As Greeno (1981) states, not all students can describe the reasons they have used their erroneous rules. We have also provided a list of erroneous rules in the Appendix.

Case Studies

Case 1.

At the beginning of the test, Kelly displayed uncertainty about the rules for adding fractions. In the first 10 problems she used three different rules for adding fractions. She multiplied the two numerators and the two denominators to obtain the corresponding parts of the answer four times and added numerators while multiplying denominators twice. She kept common denominators in the answer three times; however,

three like denominators were multiplied rather than being treated as common denominators.

During the rest of the test, a similar pattern was observed with the method of adding both the numerators and denominators becoming dominant by the end. This student added whole number parts correctly and 13 of 18 attempts to convert or reduce the form of the answer were right.

Four months later, with no intervening instruction in the addition of fractions, Kelly was asked to repeat some of the problems. In the six examples used, she always kept common denominators and added unlike denominators. When asked why she used a different denominator rule in the two types of problems, she explained, "When they aren't just the same they can't be the answer so you have to do something to them." An interesting development was discovered in converting and reducing. The fraction 5 36/6 was converted to 11 1/6, and 5 8/4 became 6 1/2. Upon questioning, the student stated that the fraction part was included because, "there needed to be something." After correctly converting 5/5 to 1, she was asked why there was no fraction part in that expression. She realized that the 11 1/6 was wrong and crossed out the fraction part. The pattern of Kelly's responses had changed and she seemed to demonstrate the use of some rules in her choice of procedures.

After this experience, the teacher worked through the problem 3 4/3 + 2 2/3 using a "picture" method in whch the student combines (adds) pictures representing 3 4/3 and 2 2/3, the student easily understood that the answer was 7. The teacher then reviewed common denominators in response to Kelly's awareness that "something" needed to be done. The student then solved the following problems:

$$\frac{1}{5} + 2\frac{5}{8}$$

$$\frac{1}{5} + \frac{1}{4}$$

$$\frac{5}{6} + \frac{1}{3}$$

$$2\frac{1}{8} + 3\frac{5}{6}$$

When Kelly repeated the 48-item test she missed only four problems. Apparently Kelly was aware of many rules for the addition of fractions but was unable to decide which procedure to use in a particular problem. In some cases, vague impressions about the way the problem should "look" prompted some of her incorrect choices. Very little teacher suggestion was needed to change her score from one of the lowest to one of the highest in the group.

Case 2.

In some cases the student's response seems to be a function of writing down each step. Mary solved the problem:

$$\frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$

but later responded erroneously:

$$\frac{1}{3} + \frac{1}{2} = \frac{1}{9}$$

In the first problem the paper included the calculations:

$$\frac{2}{6} + \frac{3}{6} = \frac{5}{6}$$

while the second example contained no corresponding work on the paper. The same student made similar errors in other problems:

$$\frac{1}{2} + 1\frac{10}{7} = \frac{7}{14} + 1\frac{20}{14} = 1\frac{27}{14}$$

but;

$$\frac{1}{5} + 2\frac{5}{3} = 2\frac{8}{15}$$

In the second example the numbers were crossed out with pencil and the denominators changed to "15", but only the change of 1 to 3 was marked in the appropriate numerator while the 5 was left unchanged. The marks obscured the original numbers of the problem. Often Mary failed to copy the whole number over into the answer when converting or reducing:

$$3\frac{6}{9} = \frac{3}{3} \quad , \quad 4\frac{6}{8} = \frac{3}{4}$$

An interesting feature that we noticed was the fact that the beginning of the test paper had more written calculations and fewer errors. The first half contained only two errors in the actual value calculated (there were some failures to reduce or convert); however, the second half of the test contained eight such errors. There was some improvement in accuracy near the end which suggests that the student was becoming more skilled at thinking through the problems without writing out every step.

Case 3.

Frequently, students may receive very poor scores on tests but still know the procedures involved. For example, Becky had only 14 problems correct on the test. From the paper we could see that all problems had been answered by adding corresponding parts until Becky realized, about half-way through the test, that that rule was erroneous. She then began to erase and correct problems that she had done earlier. Even difficult conversions and multiple equivalents were calculated correctly as she revised her answers. However, Becky did not have time to correct all of the first half and did not even attempt the second half of the test.

Four months later, Becky completed the same test with only three errors, none of which reflected the erroneous rule that she had followed the first time. Conversions, reductions, and equivalent fractions were all calculated correctly. The second time, Becky seemed much more confident about her procedures and her score was much better. It appeared that whatever flash of understanding that prompted her to revise the first test was strong enough to influence her choice of procedures months later.

Case 4.

Often a student clings to a procedure that itself allows the development of an erroneous rule. Christy's test paper exemplifies the student who always rewrites a whole number as a fraction.

$$3 = \frac{3}{1} \quad , \quad 5 = \frac{5}{1}$$

This, in itself, is not an error; but, combined with Christy's uncertainty about the procedure for adding two fractions with different denominators, it leads to problems.

The first time she was tested, she always added the denominators of two fractions:

$$1\frac{3}{9} + 2 = 1\frac{3}{9} + \frac{2}{1}$$

$$= 1\frac{5}{10} = 1\frac{1}{2}$$

Four months later Christy had learned something about common denominators and no longer added unlike denominators consistently -- in fact, she frequently calculated equivalent fractions with common denominators correctly. However, the fractions that have been formed from whole numbers are treated differently since one of the denomintors is necessarily a one. The denominators are multiplied but the numerators remain unchanged and are added. Thus:

$$1\frac{3}{9} + 2 = 1\frac{3}{9} + \frac{2}{1} = 1\frac{5}{9}$$

It seems that the most important rule that Christy remembered is the one that a whole number can be written as a fraction with a denominator of one, but she seemed not to realize the value of the resultant fraction.

The same paper provided an interesting example of the way in which a partial advance in learning can result in an incorrect answer which appears the same, but actually is much different. On the first test, following the erroneous rule of adding corresponding parts, Christy calculated:

$$2\frac{1}{2} + 4\frac{2}{4} = 6\frac{3}{6} = 6\frac{1}{2}$$

After learning how to calculate common denominators, but still unsure about the rationale for doing so, the problem was worked:

$$2\frac{1}{2} + 4\frac{2}{4} = 2\frac{2}{4} + 4\frac{2}{4}$$

$$= 6\frac{4}{8} = 6\frac{1}{2}$$

This kind of difficulty seems to stem from the inability to relate an understanding of general principles to a large body of procedural knowledge.

Case 5.

In conversation, Jenny revealed that she was always confident about the procedure needed to add fractions with like denominators but that she had never been sure about what to do with unlike denominators. After discovering that she did know how to calculate equivalent fractions, she was asked whether she could turn the problem that she was uncertain about into a problem that she knew how to solve. The solution was immediately obvious to her and seemed to make her very happy. In this case, knowledge of the relationship of known procedures resulted in understanding the process.

Case 6.

At times a student seems to develop skills according to a unique but unpredictable hierarchy. In the addition of fractions, the erroneous rule "add all corresponding parts" is viewed as basic and one of the first that must be be overcome. It is possible, however, for a student to cling to that misrule while learning other skills necessary to be successful. Barb consistently made that error on her test. She only had eight correct answers -- those which had whole numbers as one of the addends and did not contain denominators which could be added. She always found common denominators (and then proceded to add them), calculated correct equivalent fractions, and showed skill in converting improper fractions and reducing answers.

It would seem that one essential concept would enable this student to correct her errors. Four months later, Barb still added all denominators but she had given up the idea of calculating common denominators completely. Apparently that concept needed reinforcement of some kind to become a part of her rule for adding fractions. When

she had no use for common denominators, Barb neglected to do it, even though she had the procedural skill.

Case 7.

Kim had trouble converting improper fractions to mixed numbers. She did state, "it is that not ok to have the top bigger than the bottom. I don't think I know [what to do]." After deciding to try dividing the "top number" by the denominator she consistently used long division -- even when she could divide the numbers in her head. It took encouragement for her to convert 5/1 without writing down a long division sign. Problems such as 16/3 and 9/2 were classed as very difficult by Kim and the conversion of 6 19/18 became 127/18 which required extreme effort. Kim tried to convert 19/18 in her head and then add the number to 6. When she was led to do this, she could see that the answer was the same and that it had required much less calculation, but her uncertainty about such "short cuts" was obvious. It is understandable that Kim would have trouble seeing what to do with improper fractions when her habits were so linked with following complicated procedures rather than applying a meaningful principle.

Case 8.

Louis made the same error 11 times on the first test. He consistently used an erroneous rule to convert improper fractions. He correctly divided the numerator by the denominator, but he then wrote the answer with the numerator of the original fraction as the denominator of the fraction part of the answer. He always reduced the resulting fraction correctly. Four months later, Louis was tested again and no longer made the same error. Every one of the problems that he had missed for that reason was now solved correctly without assistance from the teacher. The procedural error which produced the errors on his first test, even though he used it at every opportunity, did not persist, though he could not explain why.

Case 9.

On the first test, Erica occasionally failed to calculate common denominators. The main change in her retest was in those cases in which she no longer calculated common denominators. She seemed to be unsure the first time, but the second time she had more confidence -- in an

erroneous rule! Thirty-three of her answers were identical in the two tests, most of which needed further reducing or converting to mixed numbers.

Case 10.

Ben revealed how uncertainty about a correct procedure can be reflected in the variety of solutions used by a student. When a problem did not require that a common denominator be found, he was consistent in using the correct method, but he was equally consistent in failing to follow any pattern when solving problems with unlike denominators. His rule indicated that the numerators should always be added, but the denominators might be added, multiplied, or either the first or the second chosen as the denominator of the answer. This random type of behavior did not lead to calculating equivalent fractions -- a step that Ben never took.

Four months later, Ben still had difficulty with problems that contained fractions with unlike denominators. He found the lowest common denominator more often than he did before, but he still failed to obtain equivalent fractions, even though his work on reducing and converting showed that he could have done the calculations. Apparently, his lack of understanding about the relationship of the parts of a fraction prevented his selection of a correct procedure.

Case 11.

When confronted with the problem of adding a whole number to a fraction, students generated many erroneous rules. For example, Thomas always converted a mixed number to an improper fraction before adding. He attempted to do the same thing with whole numbers, but he had one problem. The equivalent fraction of the whole number was always one, no matter what the whole number was. He did match the denominator to the other fraction in the problem; however, the numerator was always the same number as the denominator. Thus:

$$1\frac{3}{9} + 2 = \frac{12}{9} + \frac{9}{9}$$

$$= \frac{21}{9} = 2\frac{1}{3}$$

Case 12.

Eric seemed to feel that when adding a mixed number to a whole number, the fraction part should be inverted if it happened to be an improper fraction. He did not follow that erroneous rule when the addition did not involve a whole number. The whole number gave the problem very special properties for Eric.

$$3 + \frac{10}{5} = 3\frac{5}{10}$$

Case 13.

The very complicated procedure that Curtis used when adding a whole number to a fraction changed in one aspect over the eight problems of that type on the test. He seemed to be certain about one erroneous rule: the whole number should be multiplied by the denominator of the fraction and the numerator of the fraction should be added to the result. However, uncertainty about what to do with that number produced several patterns. In two cases he simply used that number as a whole number result. Other times he used it as the denominator of the answer with the numerator of the fraction as the numerator of the answer, and finally, in the last four problems, that number became the numerator of the answer with the denominator of the fraction being kept as the denominator of the answer.

Case 14.

Reese used the whole number as the numerator of a fraction to be added to another quantity. The denominator was the same as the denominator of the other fraction. This turned the whole number into a fraction that he could deal with. The denominators were common and the numerators were usually added, a correct procedure except for the fact that his original fraction was obtained in an erroneous way.

Case 15.

The confusion about the value of whole numbers extends to a kind of "double treatment" in the case of Kim's test paper. The whole number was used as a quantity to be added into the answer; however, it was also part of a fraction with a denominator of one which was then added to the other denominator. It is interesting to note that the whole number was

only added to the other whole number and not to the numerator:

$$3 + 1\frac{5}{2} = 4\frac{5}{3}$$

Case 16.

Beverly solved the problem of adding a fraction to a whole number by changing the whole number to a fraction according to a special rule. The fraction which resulted from her conversion equaled the whole number with a fraction part made up of the same number over itself. For example, 3 would be 3 3/3 by that rule. That fraction was then added to the other fraction by adding all corresponding parts.

Case 17.

Sometimes the method of calculating a number is extremely intricate. George multiplied a whole number by the denominator of the fraction that was to be added to it. Then he added the numerator to obtain the result. However, when the fraction was part of a mixed number, the process became more complicated. The first multiplication and addition were performed, then the whole number was added and finally the denominator was added:

$$1\frac{3}{9} + 2 = (2x9+3+1+9) = 31$$

Usually that number was used as a whole number answer; however in three cases it was used as the numerator of a fraction answer with the denominator of the mixed fraction as the denominator. It seemed possible that the calculation of the number was so complex that George lost track of his intended denominator -- or perhaps lost energy.

Case 18.

On the other hand, students who use the erroneous rule of adding denominators may obtain the correct answers when adding fractions to whole numbers. Steve had his only correct answers on problems of that type.

Case 19.

Mike followed the incorrect rule of converting whole numbers into their reciprocals. Thus 2 becomes 1/2. After making this conversion, he attempted to calculate common denominators, but his procedure for that was often faulty. Occasionally he multiplied the

numerators by a whole number -- even when the denominators correctly remained the same, and he seemed to have no rule for finding equivalent fractions.

Case 20.

Occasionally a student takes great care to calculate the whole number equivalent but then ignores the whole number part of the mixed fraction when adding the two. Kelly had this problem. After changing 2 to 18/9 she added it to 3/9 rather than 1 3/9 as the problem stated. This seems to indicate that her attention had been directed to one part of the procedure and that she had not generalized it to other cases.

Case 21.

The actual areas of uncertainty could be seen easily on Paula's paper. She consistently solved the problem of obtaining common denominators by using the second of two unlike denominators as the denominator of the answer. She also consistently ignored the whole number parts of mixed fractions when adding. Combining the two erroneous rules:

$$2\frac{1}{3} + 3\frac{1}{4} + \frac{2}{6} = \frac{4}{6}$$

When she was required to add fractions to whole numbers, however, she was not so certain about the procedure. Only one of the eight problems of that type followed her usual rule:

$$1\frac{3}{9} + 2 = \frac{3}{9}$$

The other problems followed a diversity of rules. In some cases, she used the whole number as the denominator of the answer, obtaining the numerator from adding the numerator and denominator of the fraction part of the other quantity. Other times she multiplied the whole number by the denominator of the fraction to get the denominator of the answer while she added the numerator and denominator of the fraction to get the numerator. Once she added the whole number and the whole number part of the mixed fraction for the numerator and kept the denominator in the

answer, and another time she simply inverted the fraction part of the mixed number:

$$2 + 1\frac{7}{3} = \frac{3}{7}$$

The sharp difference between Paula's areas of consistency and inconsistency reflected the strength of her confidence in her rules - erroneous though they may be.

Case 22.

Kim did not know how to simplify her answers. She knew that dividing and subtracting were involved in conversion in some way, but she was not sure of the procedure. In some problems she tried to divide the numerator by the denominator and use the result as the numerator and the remainder as the denominator:

$$\frac{14}{10} = \frac{1}{4}$$

Other times she seemed to convert properly and then invert:

$$7\frac{16}{8} = 7\frac{1}{2}$$

Or it could be interpreted that she subtracted the smaller from the larger and did various things with the numbers obtained:

$$3\frac{5}{9} = 3\frac{4}{5}$$

She also was inconsistent about the rule for using common denominators. Sometimes she added them and other times she kept them in the answer. The variety of her responses suggests that no procedure that she feels is "correct" has been adopted yet.

Case 23.

One erroneous rule is to combine, in some way, the values of the first addend to find the numerator of the answer and combine the values of the second addend to find the denominator. Students who used some variation of this rule had different methods of dealing with more than two

addends. Laura's method for combining values when adding mixed fractions was addition:

$$2\frac{8}{6} + 3\frac{10}{6} = (2+8+6)/(3+10+6) = \frac{16}{19}$$

When adding simple fractions, she added numerators and denominators.

When faced with three mixed fractions to add, she chose the solution of adding the sums of the numbers in each quantity together to obtain a whole number answer:

$$2\frac{1}{3} + 1\frac{2}{9} + 3\frac{1}{2}$$

$$= \frac{1}{3} + (6+12+6) = 24$$

She followed this procedure in both of the test examples of that type, though she never calulated a whole number answer in the other problems unless it was a result of conversion of an improper fraction. Apparently she was aware of the logical inconsistency of her normal procedure and felt it necessary to provide another rule. Laura's inability to convert mixed fractions no doubt contrib ted to her use of both erroneous rules.

Case 24.

Occasionally it is possible to determine from examination of the student's paper the point at which changes took place in his or her thinking, Keith's paper provides an example of that. He used a set of erroneous rules to obtain the result of adding two fractions or mixed numbers. In each of these erroneous rules, the digits in the first quantity were combined to obtain the numerator of the answer and the digits in the second quantity were combined to obtain the denominator of the answer. The rules employed were as follows:

1. In the case of a fraction, add the numerator and the denominator.

$$\frac{3}{5} + \frac{1}{5} = \frac{8}{6}$$

2. In the case of a mixed number, add the whole number part to the denominator of the fraction part. Ignore the numerator of the fraction part.

$$1\frac{4}{7} + 1\frac{12}{7} = \frac{8}{8}$$

$$\frac{1}{2} + 1\frac{10}{7} = \frac{3}{8}$$

3. Use whole numbers as they are.

$$\frac{12}{8} + 1 = \frac{20}{1}$$

These erroneous rules were used consistently for 23 problems. However, problem 24 contained, not 2, but 3 quantities to be added:

$$2\frac{1}{3} + 3\frac{1}{4} + 2\frac{2}{6} =$$

Following his rules, the student obtained the answer: $\frac{5}{7}$

Problem 25 did not follow the previous pattern nor any other common erroneous rules that could be discovered:

$$3\frac{10}{4} + 4\frac{6}{4} = \frac{15}{10}$$

However, problem 26 did adopt a common set of erroneous rules: add numerators and denominators to obtain the corresponding members of the answer fraction. Mixed fractions are treated the same -- the whole number part is ignored:

$$\frac{2}{7} + \frac{10}{12} = \frac{12}{19}$$

$$1\frac{1}{3} + 2\frac{4}{6} = \frac{5}{9}$$

Some confusion still existed in two problems which involved adding fractions to whole numbers, but the rest of the remaining problems were solved according to the new erroneous rules.

The point of change occurred at problem 24 which involved adding three mixed fractions. One interpretation of that fact would be that the student was not satisified with a rule which resulted in the fraction 5/7/8 and, therefore, moved to a rule which would give a more reasonable appearing solution to a similar problem:

$$2\frac{1}{3} + 1\frac{2}{9} + 3\frac{1}{2} = \frac{4}{14}$$

Case 25.

Converting improper fractions was often simply ignored by the students; however, that process yielded some very creative solutions when the correct procedure could not be recalled. Shawn usually solved the problem when the numerator had two digits by using the first digit of the numerator as a whole number and the second digit as the numerator of the fraction part of a mixed fraction. If the fraction already had a whole number part, the first digit was added to it. Shawn apparently did not feel that it was necessary to convert improper fractions with one digit numerators. Occasionally, the principle was applied to denominators also -- when they had two digits and the numerator only had one:

$$\frac{1}{2} + \frac{3}{8} = 1\frac{4}{0}$$

Such seemingly arbitrary manipulation within a procedure indicates that Shawn doesn't relate the process to a general principle.

Case 26.

Another student solved the first problem which required finding a common denominator as follows:

$$\frac{2}{5} + \frac{12}{8} = \frac{8}{40} + \frac{5}{40} = \frac{13}{40}$$

Apparently his erroneous rule was to use the denominator of the other fraction as the numerator when calculating the equivalent fractions.

Just two problems later the same student wrote this response:

$$2\frac{1}{2} + 4\frac{2}{4} = 2\frac{1}{2} + 4\frac{1}{2}$$

All subsequent problems involving finding common denominators and equivalent fractions were worked correctly.

The student's work on the paper strongly suggests that the first problem was difficult for the student and he could not remember the correct procedure. However, the second example was immediately obvious and it served to assist recall.

Discussion

The case studies reveal some important considerations in the analysis of errors in students' fraction addition. One of the most obvious is the fact that students' consistent erroneous rules are not the result of illogical reasoning or inferior intelligence. The students who choose procedures which lead to incorrect answers may be exhibiting the very kind of thinking that results in outstanding creative effort. Three of these capacities are: attempting to use the means which one has at hand to solve a problem, being able to see relationships between specific procedures and the needs of particular situations, and being willing to take the risk of relying on one's own ideas. It is possible that many students who make the correct procedure selection do not possess those qualities.

Various characteristics are involved in the process of achieving high scores. The student may have an excellent memory and have recently completed a course of study on the procedure being tested. The anxiety level of the high-scoring student may be low, which enables him to attend better in class as well as perform better in testing. That student may be willing to accept and practice procedures without considering the relationships involved and perhaps encounter difficulties later if his rote memorization breaks down.

The low-scoring student, on the other hand, may have worked out a complicated and intricate system which represents much thought about the relationships between conflicting elements in his experience. Many times that student is pleased with his solution and feels that it represents his best effort. The question of that student's motivation should be considered. The fact that creating and persisting in some of the individual erroneous rules was difficult and demanding should not escape notice. For such work, the student is often told that he is in error and seldom does anyone seem to recognize his effort, much less help him understand how to adapt his rule to get correct answers. His motivation for problem solving may die without encouragement and understanding. It is possible that the apathetic student who takes the easiest way through the test was once one of those whose erroneous rules were ignored.

Effective instruction demands that teachers examine the individual errors that students commit. Those students who demonstrate original and constructive thinking should receive encouragement and understanding. If the teacher can make use of knowledge concerning error patterns, the student can be given remediation in terms that he can accept and use. With that understanding and guidance, many "poor" mathematics students can learn to evaluate their own ideas and arrive at correct solutions. With such instructional practices we may be able to prevent the situation described by Papert, "It is not uncommon for intelligent adults to turn into passive observers of their own incompetence in anything but the most rudimentary mathematics"(1980,p.39).

The combination of procedures which any student follows in a complex process like addition of fractions seems to depend upon many unknown factors such as experience, personality, memory, and recognized cues; all features of the students performance should be considered, not the correct answers alone.

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Appendix

Appendix

Observed Errors in Fraction Addition Rule Application

1. Error in converting mixed fraction to an improper fraction: multiplied the whole number part by the denominator and added numerator. The denominator was not written. The result was a whole number.

$$2\frac{1}{2} + 4\frac{2}{4} = (2x2+1) + (4x4+2) = 5 + 18 = 23$$

2. Error in converting mixed fraction to an improper fraction: multiplied the whole number by the numerator and added the denominator. Used the result as the numerator of the fraction. Then applied addition procedure error by adding the numerators and the denominators.

$$2\frac{1}{2} + 4\frac{2}{4} = (2x1+2)/2 + (4x2+4)/4$$
$$= \frac{4}{2} + \frac{12}{4} = \frac{16}{6}$$
$$= 2\frac{2}{3}$$

3. Error in converting a whole number to an equivalent fraction: a whole number was converted by appending a fraction whose numerator and denominator were equal to the whole number. Then applied an addition procedure error by adding the corresponding parts.

$$\frac{12}{8} + 1 = \frac{12}{8} + 1\frac{1}{1}$$
$$= 1\frac{13}{9}, \text{ and}$$
$$3 + \frac{10}{5} = 3\frac{3}{3} + \frac{10}{5}$$
$$= 3\frac{13}{8}$$

4. Error in converting a whole number to an equivalent fraction: the whole number was replaced by a fraction whose numerator and denominator were equal to the whole number.

$$3 = \frac{3}{3} \quad \text{and} \quad 5 = \frac{5}{5}$$

5. Error in converting mixed fraction to an improper fraction: Computed the correct numerator and denominator when converting a mixed fraction to an improper fraction, but failed to remove the original whole number. Used an addition procedure error by adding corresponding parts. Applied error in converting a mixed fraction to an improper fraction by subtracting whole numbers.

$$2\frac{1}{2} + 4\frac{2}{4} = 2\frac{5}{2} + 4\frac{18}{14}$$
$$= 6\frac{23}{6} = 3\frac{5}{6}$$

6. Error in converting a whole number to a fraction: the whole number was converted to a reciprocal before adding.

$$3 + \frac{10}{5} = \frac{1}{3} + \frac{10}{5}$$

7. Error in converting mixed fraction to an improper fraction: multiplied the whole number by the numerator.

$$3\frac{5}{7} + 4\frac{6}{7} = \frac{15}{7} + \frac{24}{7}$$

8. Error in converting mixed fraction to an improper fraction: multiplied the denominator and the numerator and then added the whole number to obtain the numerator.

$$1\frac{5}{2} = (5x2+1)/2 = \frac{11}{2}$$

9. Error in converting a whole number to an equivalent fraction: the the whole number is replaced with a fraction whose numerator and whose denominator equal the denominator of the other addend.

$$1\frac{3}{9} + 2 = 1\frac{3}{9} + \frac{9}{9}$$

10. Error in converting a fraction to an improper fraction: appended a one to all fractions and then converted the resulting mixed fraction to an improper fraction. Then applied an addition procedure error by adding numerators and denominators.

$$\frac{2}{5} + \frac{12}{8} = 1\frac{2}{5} + 1\frac{12}{8}$$
$$= \frac{7}{5} + \frac{20}{8} = \frac{27}{13}$$

11. Addition procedure error: corresponding parts were added.

$$1\frac{4}{7} + 1\frac{12}{7} = 2\frac{16}{14}$$

12. Error in converting a fraction to an equivalent fraction: used the second denominator when they were different.

$$\frac{2}{5} + \frac{12}{8} = \frac{14}{8}$$

13. Addition procedure error: added denominators if they were different.

$$\frac{3}{5} + \frac{7}{5} = \frac{10}{5} \quad \text{but}$$
$$\frac{1}{3} + \frac{1}{2} = \frac{2}{5}$$

14. Addition procedure error: first cancelled factor common to the numerator of one addend and the denominator of the other addend and then multiplied denominators if different.

$$2\frac{1}{2} + 4\frac{2}{4} = \frac{5}{2} + \frac{18}{4}$$
$$= \frac{5}{1} + \frac{9}{4} = \frac{14}{4}$$

15. Addition procedure error: added numerators, but multiplied denominators.

$$\frac{3}{4} + \frac{1}{2} = \frac{4}{8}$$

16. Error in converting a whole number to an equivalent fraction: the whole number is converted by writing it as the numerator and using the denominator from the other addend.

$$1\frac{3}{9} + 2 = 1\frac{3}{9} + \frac{2}{9}$$

17. Error in finding equivalent fractions: found the least common denominator but used the original numerators.

$$2\frac{1}{2} + 4\frac{2}{4} = 2\frac{1}{4} + 4\frac{2}{4}$$
$$= 6\frac{3}{4}$$

18. Error in finding equivalent fractions: fractions with a numerator of one are converted as if the entire fraction were equal to one.

$$\frac{1}{2} + \frac{3}{8} = \frac{16}{16} + \frac{6}{16}$$

19. Error in finding equivalent fractions: multiplied the numerator by denominator of other fraction. Found least common denominator.

$$\frac{4}{15} + \frac{1}{10}$$
$$= \frac{40}{30} + \frac{15}{30}$$

20. Error in finding equivalent fractions: Sometimes added to numerator but sometimes multiplied.

$$\frac{4}{15} = (4+2)/(15x2) = \frac{6}{30}$$

but $$\frac{1}{10} = (1x3)/(10x3) = \frac{3}{30}$$

21. Error in finding equivalent fractions: used denominator of original fraction as the numerator of the new fraction.

$$\frac{2}{5} + \frac{12}{8} = \frac{5}{40} + \frac{8}{40}$$

22. Error in finding equivalent fractions: multiplied the numerator and the denominator by different numbers.

$$\frac{12}{8} = \frac{24}{40}$$

23. Error in finding equivalent fractions: multiplied numerator and denominator to get the numerator, regardless of the denominator being calculated.

$$\frac{2}{5} + \frac{12}{8} = \frac{10}{40} + \frac{96}{40}$$

24. Error in converting a whole number to an equivalent fraction: a whole number was converted by appending a fraction whose numerator and denominator were equal to the whole number.

$$5 + \frac{8}{4} = 5\frac{4}{4} + \frac{8}{4} = 5\frac{12}{4}$$

25. Addition procedure error: ignored 10 when adding.

$$\frac{15}{35} + \frac{10}{35} = \frac{15}{35} \quad \text{and}$$
$$\frac{16}{36} + \frac{10}{36} = \frac{16}{36}$$

26. Addition procedure error: found common denominator but added them.

$$\frac{1}{2} + 1\frac{10}{7} = \frac{7}{14} + \frac{34}{14}$$
$$= \frac{41}{28}$$

27. Addition procedure error: converted whole number part to fraction and added to fraction part (no error). Added denominators.

$$3\frac{5}{7} + 4\frac{6}{7}$$
$$= \frac{3}{1} + \frac{5}{7} + \frac{4}{1} + \frac{6}{7}$$
$$= \frac{18}{16} = 1\frac{1}{8}$$

28. Addition procedure error: added whole number part to numerator.

$$1\frac{3}{5} + 1\frac{8}{5} = (1+3+1+8)/5 = \frac{13}{5}$$

29. Addition procedure error: converted whole number to improper fraction then added numerators and denominators.

$$3 + \frac{10}{5} = \frac{13}{6} = 2\frac{1}{6}$$

30. Addition procedure error: converted whole number to fraction and added to denominator only.

$$3 + \frac{10}{5} = 3\frac{10}{6}$$

31. Addition procedure error: added denominators but kept the same numerator when they were the same (perhaps looking for a common numerator). This happened usually when the numerators were "1." Therefore, it might indicate difficulty with the value of one.

$$\frac{1}{3} + \frac{1}{2} = \frac{1}{5}$$

32. Addition procedure error: added one to whole number part when adding mixed numbers.

$$1\frac{2}{5} + \frac{3}{5} = 2\frac{5}{5} = 3$$
$$2\frac{8}{6} + 3\frac{10}{6} = 6\frac{18}{6} = 9$$

33. Addition procedure error: omitted whole number in adding mixed numbers.

$$2\frac{8}{6} + 3\frac{10}{6} = \frac{18}{6} = 3$$

34. Addition procedure error: when adding whole numbers to fractions, the whole number became the denominator of the answer. The numerator and denominator of the fraction were added for the numerator.

$$5 + \frac{8}{4} = \frac{12}{5}$$

$$\frac{18}{12} \div 3 = \frac{30}{3}$$

35. Error in converting a mixed fraction: a mixed fraction was converted to a whole number by multiplying the whole number by the denominator and adding the numerator. Error in converting a fraction: a fraction was changed to a whole number by adding the numerator and the denominator. The final answer was found by adding the resulting whole number to the whole number in the original problem. Error made only when adding whole number to fraction.

$$1\frac{3}{9} + 2 = 12 + 2 = 14 \text{ , and}$$

$$\frac{12}{8} + 1 = 20 + 1 = 21$$

36. Addition procedure error: to add a whole number and a fraction or mixed number, multiplied the whole number by the denominator of the fraction and added the numerator to get the denominator. The numerator was obtained by adding the whole number part of the mixed number (if there was one) to the whole number. There was no numerator in the case of adding a whole number to a fraction.

$$1\frac{3}{9} + 2 = 3/(2x9+3) = \frac{3}{21} \text{ and}$$

$$3 + \frac{10}{5} = 3x5+10 = 25$$

37. Addition procedure error: added all numbers in the first fraction to obtain the numerator and added all numbers in the second fraction to obtain the denominator.

$$2\frac{8}{6} + 3\frac{10}{6} = (2+8+6)/(3+10+6) = \frac{16}{19}$$

38. Addition procedure error: multiplied the whole number by the denominator and added the numerator. Used this as the numerator of the answer. Did the same for the parts of the second fraction and used as the denominator. The whole numbers were usually added also.

$$2\frac{8}{6} + 3\frac{10}{6} = 2 + 3 + (2x6+8)/(3x6+10)$$

$$= 5\frac{20}{28} = 5\frac{5}{7}$$

39. Addition procedure error: added numerator and denominator of first fraction to obtain the numerator and did the same for the second fraction to obtain the denominator.

$$\frac{3}{5} + \frac{1}{5} = \frac{8}{6}$$

40. Addition procedure error: added the whole number and the denominator of the first fraction to obtain the numerator of the answer. Used a similar procedure with second addend to find the denominator.

$$1\frac{4}{7} + 1\frac{12}{7} = \frac{8}{8}$$

41. Addition procedure error: when adding a fraction to a mixed number or a whole number, combined the two rules above.

$$\frac{1}{2} + 1\frac{10}{7} = \frac{3}{8} \quad , \text{ and}$$
$$\frac{12}{8} + 1 = \frac{20}{1}$$

42. Addition procedure error: multiplied numerators.

$$1\frac{4}{7} + 1\frac{12}{7} = 2\frac{48}{7} = 8\frac{6}{7}$$

43. Addition procedure error: multiplied whole numbers.

$$2\frac{8}{6} + 3\frac{10}{6} = 6\frac{18}{6}$$

44. Error in converting a mixed number: inverted the fraction part of mixed number.

$$3 + 1\frac{5}{2} = 4\frac{2}{5}$$

45. Addition procedure error: added the denominator of the first fraction to the numerator of the second fraction to obtain the numerator. Added the numerator of the first fraction to the denominator of the second fraction to obtain the denominator.

$$\frac{8}{5} + \frac{6}{5} = \frac{11}{13}$$

46. Addition procedure error: canceled common factors, then added both numerator and denominator.

$$\frac{2}{5} + \frac{12}{8} = \frac{1}{5} + \frac{12}{4}$$
$$= \frac{13}{9} = 1\frac{4}{9}$$

47. Addition procedure error: cancelled common factors and multiplied either the numerator or the denominator while adding the other.

$$\frac{2}{5} + \frac{12}{8} = \frac{1}{5} + \frac{12}{4}$$
$$= \frac{12}{9} \quad \text{but}$$

$$2\frac{1}{2} + 4\frac{2}{4} = \frac{5}{2} + \frac{18}{4}$$
$$= \frac{5}{1} + \frac{9}{4} = \frac{14}{4} = 3\frac{1}{2}$$

48. Addition procedure error: multiplied both numerator and denominator.

$$\frac{1}{4} + \frac{3}{4} = \frac{3}{16}$$

49. Addition procedure error: found common denominator and then multiplied numerators.

$$\frac{1}{3} + \frac{1}{2} = \frac{2}{6} + \frac{3}{6}$$
$$= \frac{6}{6}$$

50. Addition procedure error: found common denominator, then subtracted smaller numerator from larger.

$$\frac{1}{2} + 1\frac{10}{7} = \frac{7}{14} + 1\frac{20}{14}$$
$$= 1\frac{13}{14}$$

51. Addition procedure error: inverted second number, cancelled, and then multiplied.

$$\frac{2}{5} + \frac{12}{8} = \frac{2}{5} + \frac{8}{12}$$
$$= \frac{1}{5} + \frac{8}{6} = \frac{8}{30} = \frac{4}{15}$$
$$2\frac{1}{2} + 4\frac{2}{4} = \frac{5}{2} + \frac{18}{4}$$
$$= \frac{5}{2} + \frac{4}{8} = \frac{5}{1} + \frac{2}{18}$$
$$= \frac{10}{18} = \frac{5}{6}$$

52. Addition procedure error: cancelled fraction parts when denominators were different, then added corresponding parts.

$$\frac{1}{2} + 1\frac{10}{7} = \frac{1}{1} + 1\frac{5}{7}$$
$$= 1\frac{6}{7}$$

53. Conversion and simplification error: never converted improper fraction to mixed number.

$$\frac{9}{7} + \frac{11}{7} = \frac{20}{7}$$

54. Conversion and simplifying error: inverted improper fraction.

$$\frac{19}{10} = \frac{10}{19}$$
$$1\frac{27}{14} = 1\frac{14}{27}$$

55. Conversion and simplifying error: when converting an improper mixed fraction to a proper mixed fraction, the resulting whole number was subtracted from the original whole number.

$$2\frac{16}{14} = 1\frac{2}{16}$$

56. Conversion and simplifying error: failed to add the whole number result of conversion to the answer.

$$1\frac{3}{5} + \frac{8}{5} = 2\frac{11}{5} = 2\frac{1}{5}$$

57. Error in converting improper fraction to mixed number: if the sum of the numerators was a two digit number, the first digit was converted to a whole number and the second one was used as the numerator of the fraction part. The resulting whole number was then added to any whole number already existing in the answer. The denominators followed the same rule: they were added if they were different and were then converted as above. No conversion was done if both the added numerator and denominator had two digits. If the deonominators were the same, they were not added and were not converted regardless of the number of digits.

$$2\frac{8}{6} + 3\frac{10}{6} = 5\frac{18}{6}$$
$$= 5 + 1\frac{8}{6} = 6\frac{8}{6} \text{ , and}$$
$$3\frac{1}{6} + 2\frac{3}{4} = 5\frac{4}{10}$$
$$= 5 + 1\frac{4}{0} = 6\frac{4}{0}$$

58. Error in converting fraction to an equivalent fraction: inverted proper fraction to make improper fraction and then converted to whole or mixed number.

$$\frac{2}{6} = \frac{6}{2} = 3 \text{ , and}$$
$$\frac{5}{6} = 1\frac{1}{5}$$

59. Error in converting improper fraction: divided numerator by 2 to obtain proper fraction. Changed nothing else.

$$5\frac{18}{12} = 5\frac{9}{12} \text{ , and}$$
$$1\frac{12}{8} = 1\frac{6}{8}$$

60. Error in converting an improper fraction: faulty division.

$$5\frac{18}{6} = 7 \text{ , and } \frac{12}{4} = 1$$

61. Error in converting a mixed number to an improper fraction: added denominator and whole number parts and then multiplied by numerator to get new numerator.

$$3\frac{10}{6} = 3+6x10/6 = \frac{90}{6}$$

62. Conversion and simplifying error: after adding numerators, divided the numerator by the denominator. Ignored the remainder and used the answer as the numerator. The denominator stayed the same. Then divided the denominator by the numerator to obtain the final whole number, this time using the remainder as the numerator of the mixed fraction. The denominator always remained the same.

$$\frac{8}{5} + \frac{6}{5} = \frac{14}{5} = \frac{2}{5}$$

63. Reduction error: omitted whole number part when reducing.

$$2\frac{5}{9} + 1\frac{1}{9} = 3\frac{6}{9} = \frac{2}{3}$$

64. Reduction error: did not reduce fraction.

$$3\frac{5}{8} + 1\frac{1}{8} = 4\frac{6}{8}$$

65. Reduction error: divided numerator and denominator by different numbers. Added denominators.

$$\frac{15}{35} + \frac{10}{35} = \frac{25}{70} = \frac{5}{7}$$

66. Conversion and simplifying error: divided the larger number by the smaller. Used the numerator of the fraction as the denominator of the fraction part of the resulting mixed number.

$$\frac{4}{5} = 1\frac{1}{4} \quad , \text{ and } \quad \frac{7}{5} = 1\frac{2}{7}$$

67. Reduction error: divided all parts by the same number.

$$2\frac{16}{14} = 1\frac{8}{7}$$

68. Reduction error: subtracted denominator from numerator and used as the denominator. The numerator was always one.

$$\frac{14}{13} = \frac{1}{1} \quad , \text{ and } \quad \frac{12}{4} = \frac{1}{8}$$